A comparative study of the Hannay's angles associated with a damped harmonic oscillator and a generalized harmonic oscillator

1996 J. Phys. A: Math. Gen. 29 2607

(http://iopscience.iop.org/0305-4470/29/10/035)

J. Phys. A: Math. Gen. **29** (1996) 2607–2610. Printed in the UK

A comparative study of the Hannay's angles associated with a damped harmonic oscillator and a generalized harmonic oscillator

O V Usatenko†§, J-P Provost† and G Vallée‡

† Institut Non Linéaire de Nice, UMR CNRS 129, Université de Nice Sophia Antipolis, 1361 Route des Lucioles 06560 Valbonne, France

‡ Laboratoire de Mathématiques J A Dieudonné, URA CNRS 168, Université de Nice Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 2, France

Received 1 December 1995

Abstract. In this paper we show that a damped harmonic oscillator (the simplest dissipative physical system) is canonically equivalent to a generalized harmonic oscillator (a conservative system) even for time-dependent parameters. As a consequence, the Hannay's angles and the adiabatic invariants of the two systems appear to be the same, modulo this equivalence. This raises the question of whether this analogy can be extended to other nonlinear dissipative systems.

The Hannay's angle [1] (the classical counterpart of the Berry's geometric phase [2]), originally associated with the adiabatic evolution of classical Hamiltonian systems, has been recently extended to a large class of dynamical equations corresponding to dissipative non-Hamiltonian systems: nonlinear equations with limit cycles [3] or with more general internal symmetries [4], equations describing the dynamics of the laser [5], and so on. In this paper, we first show that one of the historical examples of Hannay's angle, namely the one associated with the generalized harmonic oscillator (GHO), can also be associated with a well known and simple dissipative system: the damped harmonic oscillator (DHO). Then we prove, more generally, that the two systems are indeed canonically equivalent, even for time-dependent parameters. Although the link between these systems has already been considered in the literature, in particular in connection with the quantization of the DHO [6], no such systematic and simple comparison of the two models exists.

The GHO and DHO dynamical systems are specified, respectively, by the Hamiltonian

$$H = \tfrac{1}{2}(xQ^2 + 2yQP + zP^2) \tag{1}$$

and by the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}) + 2\gamma\dot{q} + kq = 0. \tag{2}$$

Let us first rederive the Berry's and Hannay's results relative to the adiabatic theory of the GHO in a way which does not call for the Hamiltonian character of this model. This

§ On leave from the Department of Physics, Kharkov State University, Kharkov 310077, Ukraine.

0305-4470/96/102607+04$19.50

approach will be applied directly hereafter to the DHO equation (2). When the parameters are fixed, one easily verifies that the solution of the GHO Hamilton equations

$$\dot{Q} = yQ + zP \qquad \dot{P} = -xQ - yP \tag{3}$$

reads

$$Q = r\cos\Theta \qquad P = -\frac{r}{z}(y\cos\Theta + \omega\sin\Theta). \tag{4}$$

In (4), the angle variable Θ is given by

$$\Theta = \omega t + \alpha \qquad \left(\omega = \sqrt{xz - y^2}\right) \tag{5}$$

where α is an arbitrary constant and r is a constant depending on the energy of the system:

$$H = \tfrac{1}{2}z^{-1}\omega^2 r^2. \tag{6}$$

When the parameters vary adiabatically, the solution can still be written in the form (4), but now $r(t)$ and $\Theta(t)$ are unknown functions of time and $\dot{r}$ and $\dot{\Theta} - \omega$, instead of being equal to zero, become averaged values proportional to the time derivative of the parameters. The natural procedure to obtain these quantities is to apply the averaging method [7]. Starting from the exact equations (3), one first solves them for $\dot{\Theta}$ and $\dot{r}$ and then averages the obtained expressions with respect to the fast variable Θ. A straightforward calculation leads to the relations

$$2\frac{\dot{r}}{r} - \frac{\dot{z}}{z} + \frac{\dot{\omega}}{\omega} = 0 \tag{7}$$

$$\dot{\Theta} = \omega + \frac{y}{2\omega}\left(\frac{\dot{z}}{z} - \frac{\dot{y}}{y}\right). \tag{8}$$

Equation (7) yields the adiabatic invariant

$$I = \tfrac{1}{2}z^{-1}\omega r^2 \tag{9}$$

which is the action variable (related to the instantaneous Hamiltonian by $H = I\omega$). Equation (8) shows that the angle variable is not equal to the dynamical contribution $\int^t \omega(s)\,\mathrm{d}s$, but acquires an additional contribution (the Hannay's angle) the value of which depends on the path Γ followed by the parameters x, y and z in the parameters space. For a cyclic evolution one immediately recovers, from the 1-form $(y/2\omega)(\mathrm{d}z/z - \mathrm{d}y/y)$, the result obtained in [1],

$$\Theta_{\mathrm{H}}(\Gamma) = \iint_{S(\Gamma)} \frac{1}{4\omega^3}(z\,\mathrm{d}x \wedge \mathrm{d}y + x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x) \tag{10}$$

where $S(\Gamma)$ is any surface with boundary the loop Γ.

The DHO described by equation (2) or equivalently by

$$\ddot{q} + 2\bar{\lambda}\dot{q} + \omega_o^2 q = 0 \qquad \bar{\lambda} = \lambda + \frac{\dot{m}}{2m},\ \lambda = \frac{\gamma}{m},\ \omega_o^2 = \frac{k}{m} \tag{11}$$

can be treated in a similar way. For fixed values of the parameters (i.e. for $\bar{\lambda} = \lambda$) the solution is

$$q(t) = a\mathrm{e}^{-\lambda t}\cos(\omega t + \alpha) \qquad \left(\omega = \sqrt{\omega_o^2 - \lambda^2}\right). \tag{12}$$

In the adiabatic regime, one looks for a solution of the type

$$q = A\cos\Theta \qquad \dot{q} = -A(\lambda\cos\Theta + \omega\sin\Theta) \tag{13}$$

where A and Θ are now unknown functions of time. Then, applying the above explained averaging method to the two equations $\mathrm{d}q/\mathrm{d}t = \dot{q}$ and $\mathrm{d}\dot{q}/\mathrm{d}t = -2\bar{\lambda}\dot{q} - \omega_o^2 q$, one obtains the relations

$$2\frac{\dot{A}}{A} + \frac{\dot{m}}{m} + \frac{\dot{\omega}}{\omega} + 2\lambda = 0 \tag{14}$$

$$\dot{\Theta} = \omega - \frac{\lambda}{2\omega}\left(\frac{\dot{m}}{m} + \frac{\dot{\lambda}}{\lambda}\right) = \omega - \frac{\dot{\gamma}}{2m\omega}. \tag{15}$$

Equation (14) shows that the DHO admits, in spite of the fact that it is a dissipative system, an adiabatic invariant:

$$J = m\omega A^2 \exp\left[2\int^t \lambda(s)\,\mathrm{d}s\right]. \tag{16}$$

This invariant generalizes the quantity $m\omega^{-1}(\omega^2 q^2 + (\dot{q} + \lambda q)^2)\exp(2\lambda t)$ which is an integral of motion for fixed parameters. It also coincides with the action variable I in the absence of damping. In the presence of a constant or a slowly time-varying friction coefficient, the exponential term appears as a renormalization factor for the 'shrinking' trajectory and J still represents the area of the (closed) 'renormalized' trajectory in the space $(q, m\dot{q})$. Note that $\mathrm{d}A/A$ given by (14) is integrable and thus contains no geometrical contribution, in contradistinction to $\mathrm{d}\Theta$ given by (15). As concerns the time derivative $\dot{\Theta}_{\mathrm{H}} = \dot{\Theta} - \omega$ of the Hannay's angle of the DHO, it appears to be proportional to the derivative $\dot{\gamma}$ of the friction constant. Consequently, even in the presence of friction, the Hannay's angle remains equal to zero only if the parameters m and k vary.

Formulae (14) and (15) are very similar to (7) and (8). They are in fact strictly identical to them if one makes the following correspondence between the parameters of the two models:

$$x = k = m\omega_o^2 \qquad y = \lambda = \frac{\gamma}{m} \qquad z = \frac{1}{m}. \tag{17}$$

This suggests that the two systems (GHO and DHO) are canonically equivalent. In order to exhibit this point, let us write the Hamiltonian (1) in the form

$$H = \frac{P^2}{2m} + \lambda P Q + \frac{m\omega_o^2}{2}Q^2. \tag{18}$$

From H, one deduces the Lagrangian

$$L(Q, \dot{Q}) = \frac{m}{2}(\dot{Q}^2 - 2\lambda Q\dot{Q} - (\omega_o^2 - \lambda^2)Q^2) \tag{19}$$

and the Euler–Lagrange equation of motion

$$\ddot{Q} + \frac{\dot{m}}{m}\dot{Q} + \left(\omega_o^2 - \lambda^2 - \lambda\frac{\dot{m}}{m} - \dot{\lambda}\right)Q = 0. \tag{20}$$

In [6] this equation has been reduced to a DHO equation through a redefinition of the parameters of the GHO. Unfortunately this method does not lead to the simple physical correspondence (17) between the parameters of the two models, i.e. it does not allow the interpretation the GHO Hamiltonian as describing the motion of a particle of mass m, bound to a spring with Hooke's constant k and submitted to a friction force with coefficient γ. We propose a more natural method of comparison using a change of variables instead of parameters. Formula (16) suggests that this change of variables must be such that the Q coordinate of the GHO is the renormalized q coordinate of the DHO defined by

$$Q = q\exp\left[\int^t \lambda(s)\,\mathrm{d}s\right]. \tag{21}$$

Indeed, it is easy to verify that the Lagrangian (19) then takes the generalized Caldirola–Kanai form [8]

$$\mathcal{L}(q,\dot{q}) = \frac{m}{2}\exp\left[2\int^t \lambda(s)\,\mathrm{d}s\right](\dot{q}^2 - \omega_o^2 q^2) \tag{22}$$

and that the associated Euler–Lagrange equation of motion becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}) + 2\gamma\dot{q} + kq = 0. \tag{23}$$

This last equation clearly describes a DHO, the parameters of which are now exactly those of the Hamiltonian (18). In the Hamiltonian formalism the correspondence between the conjugate momenta P and $p = \partial\mathcal{L}/\partial\dot{q}$ reads

$$P = p\exp\left[-\int^t \lambda(s)\,\mathrm{d}s\right] = m\dot{q}\exp\left[\int^t \lambda(s)\,\mathrm{d}s\right] \tag{24}$$

and (21) and (24) appear to define a canonical transformation specified by the generating function $F(q, P, t) = qP\exp[\int^t \lambda(s)\,\mathrm{d}s]$. Therefore the trajectory of the GHO in the (Q, P) space is obtained from that of the DHO in the $(q, m\dot{q})$ space by the above-described renormalization transformation.

In conclusion to this comparative study of the DHO and GHO adiabatic behaviour, let us note that when the parameters are kept fixed the second-order equation (11) can also be written under the first-order complex form

$$\dot{z} = (\mathrm{i}\omega - \lambda)z \qquad \omega^2 = \omega_o^2 - \lambda^2 \tag{25}$$

where

$$z = q - \frac{\mathrm{i}}{\omega}(\lambda q + \dot{q}). \tag{26}$$

When the parameters vary slowly with time (keeping for simplicity the mass m constant), the right-hand side of equation (25) gains the additional term $-(\dot{\omega}/2\omega)(z - \bar{z}) - \mathrm{i}(\dot{\lambda}/2\omega)(z + \bar{z})$ which comes from definition (26) of z. The averaging procedure then suppresses the $\bar{z}$ term in the new equation for z and leads for the modulus and argument of z to two equations identical to (14) and (15). This simple way to recover the adiabatic invariant and the Hannay's angle of the DHO is interesting because equation (25) is known to be the normal form, for finite λ, of the equation for damped nonlinear oscillators. Therefore the Hannay's angles of such oscillators are related to that of the DHO. Work on this Hamiltonian approach of the adiabatic behaviour of more general dissipative nonlinear systems is in progress.

References

[1] Hannay J-H 1985 *J. Phys. A: Math. Gen.* **18** 221
Berry M V 1985 *J. Phys. A: Math. Gen.* **18** 15
[2] Berry M V 1984 *Proc. R. Soc.* A **392** 45
[3] Kepler T B, Kagan M L and Epstein I R 1991 *Chaos* **1** 455
[4] Landsberg A S 1992 *Phys. Rev. Lett.* **69** 865
[5] Todorov V Yu and Debrov V 1993 *Phys. Rev.* A **50** 878
[6] Kobe D H 1991 *J. Phys. A: Math. Gen.* **24** 2763 and references therein
[7] Arnold V I 1983 *Geometrical Methods in the Theory of Ordinary Differential Equations* (New York: Springer)
Verhulst F 1990 *Nonlinear Differential Equations and Dynamical Systems* (Berlin: Springer)
[8] Caldirola P 1941 *Nuovo Cimento* **18** 393
Kanai E 1948 *Prog. Theor. Phys.* **3** 440